

MIDTERM TEST

Date: April ... , 2009 • Duration: 90 minutes

SUBJECT: Calculus I	
Head of Department of Mathematics	Lecturer:
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INSTRUCTIONS: Students choose **FIVE** out of **8** questions. Only these will be marked. Each question carries **20** marks.

Document prohibited (except dictionaries and calculators).

Question 1. Find the following limits

(a) $\lim_{x \rightarrow 1} (2x^2 + x + 1)$, (b) $\lim_{x \rightarrow \frac{\pi}{6}} \sin(2x)$, (c) $\lim_{x \rightarrow 1} \left(\frac{1}{x-1} - \frac{2}{x^2-1} \right)$

(d) $\lim_{x \rightarrow 0} \frac{|x| \sin(x^9)}{x^9}.$

Question 2. (a) Calculate the derivative of y with respect to x if y is implicitly given by

$$\sin(xy) = y.$$

(b) Find $f'(x)$ if

$$f(x) = \frac{\sqrt[3]{1+x}}{(1-x)^3}.$$

Question 3. Let

$$f(x) := \begin{cases} x^3 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(a) [10 marks] Determine whether or not $f'(0)$ exists.

(b) [5 marks] Find $f'(x)$ (where it exists).

(c) [5 marks] Is the derivative $f'(x)$ continuous at $x = 0$?

Question 4.

(a) [7 marks] Let $y = f(x) = \frac{2}{1+e^{-x}}$. Find the equation of the tangent to the graph of the function f at the point $(0, 1)$.

(b) [7 point] Let $r(x) = f(g(h(x)))$. Given that $h(1) = 2$, $h'(1) = 4$, $g(2) = 3$, $g'(2) = 5$, and $f'(3) = 6$. Find $r'(1)$.

(c) [6 marks] Let A and B are arbitrary real numbers. Show that the function $y = Ae^{-x} + Bxe^{-x}$ satisfies the equation:

$$y'' + 2y' + y = 0.$$

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Question 5. (a) Find

$$\lim_{x \rightarrow 1} \frac{(x-1)\sqrt{2-x}}{x^2-1}.$$

(b) Show that the equation $x^3 - 5x + 1 = 0$ has at least one root.

Question 6. On the moon, if an arrow is shot upward with a velocity of 50 m/s, its height H in meters after t second is given by $H(t) = 50t - 0.083t^2$.

- (a) Find the velocity of the arrow after 10 seconds.
- (b) When will the arrow hit the moon?
- (c) With what velocity will the arrow hit the moon?
- (d) Find the maximum height reached by the arrow.

Question 7. A clothing company finds that the cost C in dollars of manufacturing x shirts is

$$C(x) = 200 + 0.8x + 0.006x^2.$$

The instantaneous rate of change of C with respect to x is called the marginal cost.

(a) Find the marginal cost when $x = 100$. What is the meaning of $C'(100)$ to the company.

(b) Find $\frac{dy}{dx}$ if $y = \sqrt{1 + \ln^2 x}$.

Question 8. Let $N(t)$ be the number of admitted students at time t of the school of Business Administration (BA) given annually from 2004 to 2008 in the following Table:

t	$N(t)$
2004	27
2005	102
2006	95
2007	116
2008	89

(a) [15 marks] Give the best estimation (up to your opinion) the value of the derivative $N'(t)$ in the years 2005, 2006, 2007.

(b) [5 marks] Could you predict the value $N(2009)$?

*** END OF QUESTION PAPER ***

SOLUTIONS TO MIDTERM EXAMINATION

CALCULUS I, April 17, 2009

(N.V. Thu, N. Dinh, J. Harris, N.N. Hai, M.D. Thanh)

Question 1.

(a)

$$\lim_{x \rightarrow 1} (2x^2 + x + 1) = 2(1^2) + 1 + 1 = 4 \quad [5 \text{ marks}]$$

(b)

$$\lim_{x \rightarrow \frac{\pi}{6}} \sin(2x) = \sin(2\pi/6) = \sin(\pi/3) = \frac{\sqrt{3}}{2} \quad [5 \text{ marks}]$$

(c) We have

$$\begin{aligned} \frac{1}{x-1} - \frac{2}{x^2-1} &= \frac{x+1}{x^2-1} - \frac{2}{x^2-1} = \frac{x-1}{x^2-1} \\ &= \frac{x-1}{(x-1)(x+1)} = \frac{1}{x+1} \end{aligned} \quad [3 \text{ marks}]$$

Thus

$$\lim_{x \rightarrow 1} \left(\frac{1}{x-1} - \frac{2}{x^2-1} \right) = \lim_{x \rightarrow 1} \frac{1}{x+1} = \frac{1}{1+1} = \frac{1}{2} \quad [2 \text{ marks}]$$

(d) **METHOD 1**

Suppose first that $0 < x \leq \pi/4$. Using the unit circle we have $0 < \sin x \leq x$ which implies that

$$0 < \frac{\sin x}{x} \leq 1.$$

Similarly, for $-\pi/4 \leq x < 0$. So for $0 < |x| \leq \frac{\pi}{4}$,

$$0 < \frac{\sin x}{x} \leq 1. \quad [1 \text{ mark}]$$

Consequently, for $0 < |x| \leq (\frac{\pi}{4})^{1/9}$,

$$0 < \frac{\sin(x^9)}{x^9} \leq 1. \quad [1 \text{ mark}]$$

So for $0 < |x| \leq (\frac{\pi}{4})^9$,

$$0 < |x| \frac{\sin(x^9)}{x^9} \leq |x|. \quad [1 \text{ mark}]$$

We know $\lim_{x \rightarrow 0} |x| = 0$. So by the Squeeze theorem,

$$\lim_{x \rightarrow 0} \frac{x \sin(x^9)}{x^9} = 0 \quad [2 \text{ marks}]$$

METHOD 2 We can write

$$\lim_{x \rightarrow 0} \frac{|x| \sin(x^9)}{x^9} = \lim_{x \rightarrow 0} |x| \cdot \lim_{x \rightarrow 0} \frac{\sin(x^9)}{x^9}. \quad [1 \text{ mark}]$$

We know

$$\lim_{x \rightarrow 0} |x| = 0. \quad [1 \text{ mark}]$$

Now consider $\lim_{x \rightarrow 0} \frac{\sin(x^9)}{x^9}$. If $u = x^9$ then $u \rightarrow 0$ iff $x \rightarrow 0$. So,

$$\lim_{x \rightarrow 0} \frac{\sin(x^9)}{x^9} = \lim_{u \rightarrow 0} \frac{\sin u}{u} = 1. \quad [2 \text{ marks}]$$

Hence

$$\lim_{x \rightarrow 0} \frac{|x| \sin(x^9)}{x^9} = 0 \cdot 1 = 0. \quad [1 \text{ mark}]$$

Question 2.

a) Taking derivatives of both sides with respect to x we have

$$\frac{d}{dx} \sin(xy) = \frac{dy}{dx}$$

or

$$\cos(xy)(xy)' = y' \quad [3 \text{ marks}]$$

$$\Leftrightarrow (y + xy') \cos(xy) = y' \quad [3 \text{ marks}]$$

So

$$y'(1 - x \cos(xy)) = y \cos(xy) \quad [2 \text{ marks}]$$

Therefore

$$y' = \frac{dy}{dx} = \frac{y \cos(xy)}{1 - x \cos(xy)} \quad [2 \text{ marks}]$$

b) **METHOD 1**

Taking logarithms of both sides, we have (for $-1 < x < 1$)

$$\ln f(x) = \ln \left(\frac{\sqrt[3]{1+x}}{(1-x)^3} \right) = \frac{1}{3} \ln(1+x) - 3 \ln(1-x) \quad [3 \text{ marks}]$$

Differentiating both sides gives

$$(\ln f(x))' = \left(\frac{1}{3} \ln(1+x) - 3 \ln(1-x) \right)'$$

or

$$\frac{f'(x)}{f(x)} = \frac{1}{3(1+x)} + \frac{3}{1-x} \quad [\text{LHS: 2 marks, RHS: 3 marks}]$$

Thus,

$$f'(x) = f(x) \left(\frac{1}{3(1+x)} + \frac{3}{1-x} \right) = \frac{\sqrt[3]{1+x}}{(1-x)^3} \left[\frac{1}{3(1+x)} + \frac{3}{1-x} \right]. \quad [2 \text{ marks}]$$

METHOD 2

By quotient rule and chain rule,

$$\begin{aligned} f'(x) &= \frac{(1-x)^3 (\sqrt[3]{1+x})' - \sqrt[3]{1+x} [(1-x)^3]'}{(1-x)^6} \quad [4 \text{ marks}] \\ &= \frac{(1-x)^3 \frac{1}{3} (1+x)^{-\frac{2}{3}} + \sqrt[3]{1+x} \cdot 3(1-x)^2}{(1-x)^6} \quad [4 \text{ marks}] \\ &= \frac{\frac{1}{3}(1-x)(1+x)^{-\frac{2}{3}} + 3\sqrt[3]{1+x}}{(1-x)^4} \left(= \frac{\sqrt[3]{1+x}}{(1-x)^3} \left[\frac{1}{3(1+x)} + \frac{3}{1-x} \right] \right). \quad [2 \text{ marks}] \end{aligned}$$

Question 3.

$$f(x) := \begin{cases} x^3 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(a) When $x \neq 0$,

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} x^2 \sin \frac{1}{x}. \quad [4 \text{ marks}]$$

Since

$$0 \leq |x^2 \sin \frac{1}{x}| \leq x^2,$$

we get

$$\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = \lim_{x \rightarrow 0} x^2 = 0. \quad [4 \text{ marks}]$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0,$$

which ensures that $f'(0)$ exists and $f'(0) = 0$. [2 marks]

(b) Find $f'(x)$, $x \in \mathbb{R}$.

- If $x = 0$ then by (a), $f'(0) = 0$. [1 mark]
- When $x \neq 0$ then $f(x) = x^3 \sin \frac{1}{x}$ which is differentiable. Its derivative

is:

$$\begin{aligned} f'(x) &= 3x^2 \sin \frac{1}{x} + x^3 \left(-\frac{1}{x^2}\right) \cos \frac{1}{x} \\ &= x \left(3 \sin \frac{1}{x} - \cos \frac{1}{x}\right). \end{aligned} \quad [3 \text{ marks}]$$

Thus,

$$f'(x) = \begin{cases} x(3 \sin \frac{1}{x} - \cos \frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \quad [1 \text{ mark}]$$

Question 4.

(a) Using the chain rule,

$$f'(x) = \frac{(-1)2}{(1 + e^{-x})^2} \cdot (e^{-x})' = \frac{2e^{-x}}{(1 + e^{-x})^2}. \quad [3 \text{ marks}]$$

So

$$f'(0) = \frac{2}{(1 + 1)^2} = \frac{1}{2}. \quad [2 \text{ marks}]$$

So the tangent line has gradient $m = \frac{1}{2}$.

Using $y - y_1 = m(x - x_1)$, the equation of the tangent line is

$$\begin{aligned} y - 1 &= \frac{1}{2}(x - 0) \\ y &= \frac{1}{2}x + 1 \end{aligned} \quad [2 \text{ marks}]$$

(b) Since $r(x) = f(g(h(x)))$, we have

$$r'(x) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x), \quad [3 \text{ marks}]$$

and hence,

$$r'(1) = f'(g(h(1))) \cdot g'(h(1)) \cdot h'(1). \quad [1 \text{ mark}]$$

So

$$\begin{aligned} r'(1) &= f'(g(2)) \cdot g'(2) \cdot h'(1) \\ &= f'(3) \cdot 5 \cdot 4 = 6 \cdot 5 \cdot 4 = 120. \end{aligned} \quad [3 \text{ marks}]$$

(c) We have

$$y' = -Ae^{-x} - Bxe^{-x} + Be^{-x} = e^{-x}(B - A - Bx), \quad [2 \text{ marks}]$$

$$y'' = e^{-x}(-2B + A + Bx). \quad [2 \text{ marks}]$$

Therefore,

$$\begin{aligned} y'' + 2y' + y &= e^{-x}(-2B + A + Bx) + 2e^{-x}(B - A - Bx) + Ae^{-x} + Bxe^{-x} \\ &= 0, \end{aligned}$$

which means that the function $y = Ae^{-x} + Bxe^{-x}$ satisfies the equation $y'' + 2y' + y = 0$. [2 marks]

Question 5.

(a) We have

$$\lim_{x \rightarrow 1} \frac{(x-1)\sqrt{2-x}}{x^2-1} = \lim_{x \rightarrow 1} \frac{(x-1)\sqrt{2-x}}{(x-1)(x+1)} \quad [5 \text{ marks}]$$

$$= \lim_{x \rightarrow 1} \frac{\sqrt{2-x}}{x+1} \quad [2 \text{ marks}]$$

$$= \frac{1}{2}. \quad [3 \text{ marks}]$$

(b) Let $f(x) = x^3 - 5x + 1$.

Since f is a polynomial, it is continuous on $\mathbb{R}$. [1 marks]

We have

$$f(0) = 1 > 0 \quad \text{and} \quad f(1) = -3 < 0. \quad [4 \text{ marks}]$$

Thus, by the Intermediate Value Theorem [2 marks], there is a point c between 0 and 1 such that $f(c) = 0$ [2 marks]. This means that c is a root of $f(x)$. So the equation $x^3 - 5x + 1 = 0$ has at least one root. [1 mark]

Question 6.

- (a) $v = \frac{dH}{dt} = 50 - 1.66t$ [3 marks]
 After 10 seconds, $v(10) = 50 - 16.6 = 33.4$ [1 mark]
 So the velocity of the arrow is 33.4 m/s. [1 mark for units]
- (b) The arrow hits the moon when $H = 0$. [1 mark]
 $H(t) = t(50 - 0.83t) = 0$ has solutions $t_1 = 0$ and $t_2 = \frac{50}{0.83} = 60.24$ [2 marks]
 t_1 is when the arrow is fired. So the required solution is t_2 . [1 mark]
 So the arrow hits the moon after 60.2 seconds. [1 mark for units]
- (c) From part (a), $v = 50 - 1.66t$.
 So when the arrow hits the moon, $v = v(60.24) = 50 - 1.66(60.24) = -50$. [3 marks]
 So the velocity is -50 m/s, i.e. 50 m/s downwards. [2 marks]
- (d) At maximum height, $v = \frac{dH}{dt} = 0$. [1 mark]
 That is when $v = 50 - 1.66t = 0$, $t = \frac{50}{1.66} = 30.1$. [1 mark]
 Then, $H(30.1) = 50(30.1) - 0.83(30.1)^2 = 753$. [2 marks]
 So the maximum height reached is 753 m. [1 mark for units]

Question 7.

- (a) Using the Chain Rule we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(1 + \ln^2 x)^{\frac{1}{2}} = \frac{1}{2}(1 + \ln^2 x)^{-\frac{1}{2}} \frac{d}{dx}(1 + \ln^2 x) \quad [4 \text{ marks}] \\ &= \frac{1}{2}(1 + \ln^2 x)^{-\frac{1}{2}} (2 \ln x) \frac{d}{dx} \ln x \quad [3 \text{ marks}] \\ &= \frac{\ln x}{x\sqrt{1 + \ln^2 x}} \quad [3 \text{ marks}]. \end{aligned}$$

- (b) The marginal cost is $C'(x) = 0.8 + 0.012x$ [3 marks]
 When $x = 100$, we have $C'(100) = 0.8 + 1.2 = 2$ [2 marks]
 So the marginal cost is \$2 per shirt. [2 marks for units]

$C'(x)$ represents the rate of change of the total production cost with respect to quantity produced. $C'(100) = 2$ means that at a production level of 100 shirts, the production cost is increasing at a rate \$2 /shirt. So the cost of producing the 101th shirt is approximately \$2. [3 marks]

Question 8.

(a) $N'(t)$ is the rate of increase of $N(t)$. Let us first consider $N'(2005)$.

$$N'(2005) = \lim_{h \rightarrow 0} \frac{N(2005 + h) - N(2005)}{h}$$

So for small values of h ,

$$N'(2005) \approx \frac{N(2005 + h) - N(2005)}{h}.$$

For $h = 1$, we get

$$N'_+(2005) \approx \frac{N(2006) - N(2005)}{1} = -7$$

(This is the average rate of increase between 2005 and 2006.)

For $h = -1$, we have

$$N'_-(2005) \approx \frac{N(2004) - N(2005)}{-1} = 75$$

which is the average rate of increase between 2004 and 2005.

We get more accurate approximation if we take the average of these two values:

$$N'(2005) \approx \frac{1}{2}(75 - 7) = 34.$$

This means that in 2005 the number of students admitted was increasing at a rate of about 34 people/year.

Similar calculations for $t = 2006$ and $t = 2007$ gives the following values.

Table of approximate values for the derivative $N'(t)$.

t	$N'(t)$
2005	34
2006	7
2007	-3

[Suggested markscheme: Value for one year correct, with appropriate explanation: 7 marks. Value for other two years: 4 marks each. If students only calculate values from one direction, maximum 8 marks.]

(b) We observe that $N_-(2008) \approx \frac{N(2007)-N(2008)}{-1} = -27$ is a negative number. So we can expect the number of BA students admitted this summer to be lower than the number admitted last summer.

[Predict $N(2009) < N(2008)$: 2 marks. Explanation: 3 marks.]